

Perturbations of an exact solution for 2+1 dimensional critical collapse

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We find the perturbation spectrum of a family of spherically symmetric and continuously self-similar (CSS) exact solutions that appear to be relevant for the critical collapse of scalar field matter in 2+1 spacetime dimensions. The rate of exponential growth of the unstable perturbation yields the critical exponent. Our results are compared to the numerical simulations of Pretorius and Choptuik and are inconclusive: We find a CSS solution with exactly one unstable mode, which suggests that it may be the critical solution, but another CSS solution which has three unstable modes fits the numerically found critical solution better.

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I. INTRODUCTION

Critical gravitational collapse, as first found by Choptuik [1], has been studied in many systems [2]. For the most part, these studies have been numerical: generally the critical solution has not been found in closed form. However, gravitation in 2+1 dimensions is a much simpler system than in 3+1 dimensions. Thus one might hope to find a closed form solution of 2+1 critical collapse. Pretorius and Choptuik [3] performed numerical simulations of 2+1 critical gravitational collapse with a massless, minimally coupled scalar field and a cosmological constant. A numerical treatment of this system has also been done by Husain and Olivier.[4] One of us [5] found a closed form continuously self-similar (CSS) solution of the 2+1 Einstein-scalar equations that agrees with the work of Ref. [3]. Other closed form solutions of these equations have also been treated[6, 7] and the approach to the singularity in this system has been analyzed.[8]

A critical solution, when perturbed, should have exactly one unstable mode, which grows as e^{kT} for some constant k , where T is a coordinate such that $\partial/\partial T$ is the homothetic vector of the CSS critical solution. In the collapse of a one parameter family of initial data, a quantity Q with dimension (length)^s obeys a scaling relation $Q \propto |p - p^*|^{\gamma s}$ where p is the parameter and p^* is its critical value. The quantities k and γ are related by $\gamma = 1/k$. Thus by treating perturbations of a critical solution, one could find k and compare to the numerical value of γ found in simulations of near critical collapse. Such a perturbation treatment was begun in [5] but was not completed due to questions about appropriate boundary conditions to impose on the perturbations.

In this paper, we complete the perturbation treatment begun in [5]. Throughout we use double null coordinates rather than the Bondi coordinates used in [5]. This tends to clarify the issue of boundary conditions for the perturbations. Section II presents the field equations and background solution in double null coordinates. The perturbations are treated in section III. Conclusions are presented in section IV.

II. FIELD EQUATIONS AND BACKGROUND SOLUTION

The simulations of Ref. [3] used the Einstein-scalar equation with cosmological constant. However, in the approach to the critical solution, the cosmological term becomes negligible. Therefore we approximate the Einstein equations as

$$R_{ab} = 4\pi\nabla_a\phi\nabla_b\phi. \quad (1)$$

We consider an axisymmetric 2+1 metric in double null coordinates, which takes the form

$$ds^2 = 2e^{2A}dudw + B^2d\theta^2, \quad (2)$$

where A and B are functions of u and w . We choose u at the origin to be proper time from the singularity, but place no restrictions on w . For a metric of the form (2), the Einstein-scalar equations (1) become

$$2B \frac{\partial^2 \phi}{\partial u \partial w} + \frac{\partial B}{\partial u} \frac{\partial \phi}{\partial w} + \frac{\partial B}{\partial w} \frac{\partial \phi}{\partial u} = 0, \quad (3)$$

$$\frac{\partial^2 B}{\partial u \partial w} = 0, \quad (4)$$

$$2 \frac{\partial B}{\partial w} \frac{\partial A}{\partial w} - \frac{\partial^2 B}{\partial w^2} = 4\pi B \left(\frac{\partial \phi}{\partial w} \right)^2, \quad (5)$$

(plus two additional components of the field equations which are redundant). The quantity A is fixed at the origin by the requirement that u be proper time there.

To present the background solution, it is helpful to introduce coordinates T and y given by $-u = e^{-T}$ and $y = w^n / \sqrt{-u}$ where n is a constant. There is a family of regular CSS solutions parameterized by a positive integer n (the q of Ref. [5]), with

$$\bar{B} = \frac{1}{2} e^{-T} (1 - y^2), \quad (6)$$

$$\bar{\phi} = c \left[T - 2 \ln \left(\frac{1+y}{2} \right) \right], \quad (7)$$

$$\bar{A} = \frac{1}{2} \ln n + \left(1 - \frac{1}{2n} \right) \left[-\frac{T}{2} + 2 \ln \left(\frac{1+y}{2} \right) \right]. \quad (8)$$

Here, the constant c is given by $c = \pm \sqrt{(2n-1)/8\pi n}$ and a bar denotes a background quantity. The solution with $n = 4$ appears to be a good fit to the critical solution found by Pretorius and Choptuik [3] in numerical collapse simulations.

The spacetime is CSS with homothetic vector $\partial/\partial T$. The log-scale coordinate T used here is the same as that defined in Ref. [5]. The self-similarity coordinate y defined here is related to the coordinate R defined in Ref. [5] by $y^2 = 1 - 2R$. However, this simple relation between y and R does not hold when the spacetime is perturbed. Note that in these coordinates the origin is at $y = 1$ while the past light cone of the singularity is at $y = 0$. Though we have introduced the coordinates T and y for convenience, the criterion for smoothness is that A , B and ϕ be smooth functions of u and w . The background solution is smooth provided that n is a positive integer.

To study the global structure of the background spacetimes we make another coordinate change $y \equiv x^n$. The metric becomes

$$ds^2 = e^{-2T} \left[\left(\frac{1+x^n}{2} \right)^{4(1-\frac{1}{2n})} (2n dx - x dT) dT + \frac{1}{4} (1 - x^{2n})^2 d\theta^2 \right]. \quad (9)$$

The maximal extension of the spacetime is provided by this metric with $-\infty < T < \infty$ and $-1 \leq x \leq 1$. The Ricci scalar is proportional to e^{2T} , and so $T = \infty$ is a curvature singularity. The regular center is at $x = 1$, and the past lightcone is at $x = 0$. Lines of constant x (trajectories of the homothetic vector field) are timelike for $x > 0$ and spacelike for $x < 0$. The rings of constant x and T are closed trapped surfaces for $x < 0$, and so $x = 0$ can be interpreted as an apparent horizon.

Note that the $x < 0$ region of the spacetime contains trapped surfaces, while a critical solution (since it lies on the boundary of those spacetimes with and those without trapped surfaces) cannot itself contain a trapped surface. Thus it is only the $x > 0$ part of the spacetime that matches the corresponding part of the numerical critical solution. That part of the critical solution that is outside the past light cone of the singularity is not given by our background solution.

In 2+1 dimensions, the cosmological constant is also necessary for black hole formation, and this goes beyond the approximation $\Lambda = 0$ here (and beyond our current understanding). In a related complication, in the presence of a cosmological constant, a black hole will always eventually form and capture all the scalar field matter. The threshold behaviour and mass scaling studied by Pretorius and Choptuik applies only to prompt black hole formation by a scalar field pulse that has not yet reached the AdS timelike null-infinity.

III. PERTURBATIONS

We now consider perturbations of the background solution. Here we use a δ to denote a perturbed quantity. Any perturbation mode is of the form

$$\delta B = e^{(k-1)T} b(y), \quad (10)$$

$$\delta \phi = e^{kT} H(y), \quad (11)$$

$$\delta A = e^{kT} a(y), \quad (12)$$

where k is a constant. The perturbation grows as the singularity is approached if $k > 0$.

These perturbations must satisfy the linearized versions of equations (3-5) which we now solve in turn. Substituting the ansatz (10) into the perturbation of equation (4), we find

$$y b'' + (2k - 1) b' = 0. \quad (13)$$

The general solution is

$$b = c_0 + c_1 (1 - y^{2-2k}), \quad (14)$$

for arbitrary constants c_0 and c_1 . We may also ask which infinitesimal gauge transformations $x^\mu \rightarrow x^\mu + \xi^\mu$ create metric perturbations of the form (10-12). It turns out that they form a 2-parameter family with the same parameters c_0 and c_1 . The perturbation b is therefore pure gauge. In order to study regularity at the origin, we introduce Cartesian coordinates $t = (u - w^{2n})/2$ and $\bar{r} = (-u - w^{2n})/2$. The background metric then approaches the usual Minkowski form at the origin. All metric coefficients, and the scalar field ϕ , are regular at $\bar{r} = 0$, and are even functions of \bar{r} . The gauge transformation ξ^μ takes the form

$$\xi^t = -[(c_0 + c_1)(-t + \bar{r})^{1-k} + c_1(-t - \bar{r})^{1-k}], \quad (15)$$

$$\xi^{\bar{r}} = (c_0 + c_1)(-t + \bar{r})^{1-k} - c_1(-t - \bar{r})^{1-k}. \quad (16)$$

This is a regular vector field at $r = 0$ if and only if $c_0 = 0$, and we assume this from now on. We are left with a family of linear perturbation gauges parameterized by c_1 . We shall later fix c_1 by further regularity considerations.

The perturbation of equation (3) now becomes

$$\frac{1}{2} y (1 - y^2) H'' + [k - (k + 1) y^2] H' - k y H = \frac{2cc_1}{(1 + y)^2} [k(1 + y) - y + y^{1-2k}(1 - k(1 + y))]. \quad (17)$$

The solution of this equation that is regular at the origin ($y = 1$) is

$$H = -\frac{2cc_1}{1 + y} (1 + y^{1-2k}) + c_2 F(k, 1/2, 1, 1 - y^2), \quad (18)$$

where c_2 is a constant and F is a hypergeometric function. When $c_2 = 0$, the perturbation simply results from applying an infinitesimal coordinate transformation to the background and is therefore pure gauge.

We now address the issue of smoothness of the perturbation at $y = 0$, the past light cone of the singularity. We first consider the particular cases $k = 1$ and $k = 1/2$. In the case $k = 1$, we have $F(1, 1/2, 1, 1 - y^2) = 1/y$. Therefore regularity at $y = 0$ requires that $H = 0$ for the case $k = 1$. In the case $k = 1/2$, we have $F(1/2, 1/2, 1, 1 - y^2) = (2/\pi) K(\sqrt{1 - y^2})$ where K is an elliptic integral. The quantity $K(\sqrt{1 - y^2})$ diverges at $y = 0$. Since the first term on the right hand side of equation (18) is regular at $y = 0$ for $k = 1/2$, it follows that for $k = 1/2$ a regular perturbation must have $c_2 = 0$, which is pure gauge.

Now we consider general values of k . Using an identity for hypergeometric functions (section 9.13 of [9]) we have

$$F(k, 1/2, 1, 1 - y^2) = \frac{\Gamma(\frac{1}{2} - k)}{\sqrt{\pi} \Gamma(1 - k)} F(k, \frac{1}{2}, \frac{1}{2} + k, y^2) + \frac{\Gamma(k - \frac{1}{2})}{\sqrt{\pi} \Gamma(k)} y^{1-2k} F(1 - k, \frac{1}{2}, \frac{3}{2} - k, y^2). \quad (19)$$

Here $\Gamma(x)$ denotes the gamma function. This formula applies except in the case where k is half of an odd integer. In that case, one of the two Γ -functions has a pole. The correct formula then involves a $\ln y$ term, and is therefore not regular at $y = 0$. All odd half-integer values of k must therefore be ruled out.

The hypergeometric functions on the right hand side of equation (19) are power series in y^2 and since $y \propto w^n$ they are power series in w . However, due to the expression y^{1-2k} the right hand side of equation (19) will not consist of

integer powers of w unless $k = m/(2n)$ for some integer m . We now consider the possible values of m . The cases $m = n$ ($k = 1/2$), $m = 2n$ ($k = 1$) and $m = (2p - 1)n$ for p integer (k odd half-integer) have already been ruled out. In considering the other cases, it is helpful to introduce the quantity $c_3 \equiv c_2 \Gamma(k - \frac{1}{2}) / (\sqrt{\pi} \Gamma(k))$. The solution for $H(x)$ near $x = 0$ is then

$$\begin{aligned} H &= -\frac{2cc_1}{1+x^n} (1 - x^{n-m}) + O(1) + c_3 [x^{n-m} + O(x^{3n-m})] \\ &= (c_3 - 2cc_1)x^{n-m} + 2cc_1x^{2n-m} + O(x^{3n-m}) + O(1). \end{aligned} \quad (20)$$

For $m < n$ these are all positive powers of x . For $n < m < 2n$, we have one negative power, but it is canceled when $c_1 = c_3/(2c)$. This means that there is only this one gauge choice c_1 in which this perturbation is regular. For $m > 2n$ we have a negative power x^{2n-m} which cannot be canceled, and we must therefore have $m < 2n$.

We now consider the perturbation of equation (5) which (after some straightforward but tedious algebra) becomes

$$\begin{aligned} ny \frac{da}{dy} &= -\frac{1}{(1+y)^2} \left[c_1 k(1-2n)(1-y^2)y^{-2k} + c_1 2nk(1-k)(1+y)^2 y^{-2k} \right. \\ &\quad \left. + c_1(2n-1)(y-y^{1-2k}) + (1-2n)(1-y^2)(1+y) \frac{c_2}{2c} F' \right]. \end{aligned} \quad (21)$$

Here F' is an abbreviation for $(d/dy)F(k, 1/2, 1, 1-y^2)$.

Regularity of the perturbation imposes the condition that the quantity inside square brackets in equation (21) vanishes at $y = 0$. Therefore, for each term we need only consider that part of it that does not automatically vanish in this limit. We first consider the case where $m < n$. Then, y^{1-2k} vanishes at $y = 0$ but y^{-2k} does not. Expanding the quantity inside square brackets in equation (21) to the appropriate order we find

$$ny \frac{da}{dy} = -\frac{1}{(1+y)^2} \left[c_1 k(1-2nk)y^{-2k} + \frac{c_3}{2c}(1-2n)(1-2k)y^{-2k} + o(y) \right]. \quad (22)$$

Here $o(y)$ denotes a quantity that vanishes when $y = 0$. In general, the constant c_3 can be chosen so that the quantity inside square brackets in equation (22) is $o(y)$. However, for $2nk = 1$ (*i.e.* $m = 1$) the choice of c_3 that does this is $c_3 = 0$ and thus the perturbation is pure gauge. We then arrive at the result that a physical perturbation cannot have $m = 1$.

Now consider the case $n < m < 2n$. Then both y^{-2k} and y^{1-2k} are nonvanishing as $y \rightarrow 0$. In addition, we have the restriction that $c_3 = 2cc_1$. Then expanding the quantity inside square brackets in equation (21) to the appropriate order we find

$$ny \frac{da}{dy} = \frac{c_1(k-1)}{(1+y)^2} (2nk+1-2n) [y^{-2k} + 2y^{1-2k} + o(y)] \quad (23)$$

Thus, the only way to obtain regularity of the perturbation is to have $2nk = 2n - 1$ or in other words $m = 2n - 1$. Note that there is a mode with $m = 2n - 1$ only when this value is greater than n , that is for $n > 1$.

Regularity at the origin $y = 1$ does not give rise to further regularity conditions beyond the ones already discussed.

IV. CONCLUSIONS

Perturbation modes depend on T as e^{kT} . As a result of imposing regularity conditions, we have found the discrete spectrum of growing perturbations $k = m/(2n)$, where m is an integer with the following restrictions: $m > 1$ and either $m = 2n - 1$, or $m < n$. This means that the CSS background solution with index n has $n - 1$ unstable ($k > 0$) modes. In particular, the solution with $n = 1$ has no unstable modes. The $n = 2$ solution has one unstable mode with $m = 3$. The solutions with $n > 2$ all have one unstable mode with $m = 2n - 1$ and $n - 2$ additional unstable modes with $1 < m < n$.

We now compare our results with the numerical results of Ref. [3]. The critical solution should be the one with one unstable mode. Thus our analysis indicates that the critical solution is the one with $n = 2$ and therefore that $k = 3/4$. In contrast, the analysis of curvature scaling in [3] indicates that $\gamma = 1.2 \pm 0.05$ which corresponds to $k = 0.83 \pm 0.04$. Furthermore, direct comparison between numerical and analytic critical solutions indicates a much better fit with $n = 4$ than with $n = 2$. The $n = 4$ solution has an unstable mode with $k = 7/8 = 0.875$ which is in agreement with the k of Ref. [3]. However, our analysis indicates that it also has modes with $k = 1/4$ and $k = 3/8$

and therefore this solution cannot be the critical solution. Similarly, the $n = 3$ solution has an unstable mode with $k = 5/6$ which corresponds to a γ of exactly 1.2 but it also has a unstable mode with $k = 1/3$ and therefore it seems that it cannot be the critical solution.

Alternatively, it may be that some additional condition needs to be imposed on the perturbations due to the fact that the past light cone of the singularity is an apparent horizon. Recall that the critical solution is only equal to our background solution within the past light cone of the singularity and must be something else outside the past light cone in order to avoid trapped surfaces. Thus this critical collapse situation differs from the usual one, and it may be that smoothness of the perturbation is not a sufficiently strong condition to impose. Such an additional condition might eliminate some of the modes that we have found. Some light could be shed on this issue by further numerical treatment of the original critical collapse problem. In particular, such a treatment could find the growing mode and compare it to the modes found in our perturbative treatment.

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